Tensor Decomposition to Capture Spatiotemporal Patterns of Coupled Oscillator and Opinion Dynamics

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1 Introduction

Systems of coupled oscillators or individuals interacting on networks can exhibit rich emergent behaviors such as synchronization, phase locking, and consensus or polarization of opinions [1,2,3]. Such emergent behavior depends on network topology, the governing dynamics, and the initial condition, often in a highly non-linear way. Understanding patterns of how such large-scale behavior emerges is of critical importance, which has led to numerous applications in the coordination of fleets of autonomous vehicles, information fusion, wireless sensor networks, distributed computing systems, and modeling formation of public opinions in social networks [4,5].

There have been recent works utilizing machine-learning-based methods to better understand how various large-scale behavior emerges in complex systems, which is often mathematically intractable without special symmetries [6,7]. In this work, we propose to utilize an interpretable unsupervised feature extraction method of *nonnegative CP tensor decomposition* (NCPD) [8] to **extract key spatiotemporal patterns of emergence at subgraph level.** That is, we seek to learn a few snapshots of dynamics on certain subgraph patterns and their temporal evolution by using NCPD. Tensor decomposition techniques have been widely used in various applications including learning spatiotemporal transcriptomics of human brain [9], tomography [10], data completion, and text mining [11], but never for learning spatiotemporal patterns in complex systems.

We study the *firefly cellular automata* (FCA) [12] of discrete pulse-coupled oscillators and *Hegselmann-Krause* (HK) [13] model for opinion dynamics on synthetic networks generated by the Newman-Watts-Strogatz (NWS) [14] model and the Barabási–Albert (BA) networks [15]. The code for the simulations and results in this work is provided in https://github.com/agamgoy-research/NCPD-Dynamics.

2 Method and Results

Tensor encoding of complex systems. In order to make use of nonnegative CP decomposition on tensor data, we first encode our network-dynamic data as a *color-coded adjacency tensor* (CCAT) as follows.

Suppose we are given a pair of graph $G = (V, E)$ and observed dynamics $(X_t)_{0 \le t < T}$, which are assumed to have evolved according to some coupled oscillator model or opinion dynamics model, where $X_t : V \to \mathbb{Z}/\kappa\mathbb{Z}$. We can then represent the pair $(G, (X_t)_{0 \le t < T})$ of input data by a tensor $\mathbf{X}$ of shape $k \times k \times T$, where each slice $\mathbf{X}[:,:,t]$ represents the dynamics configuration X_t imposed onto the graph topology G as follows

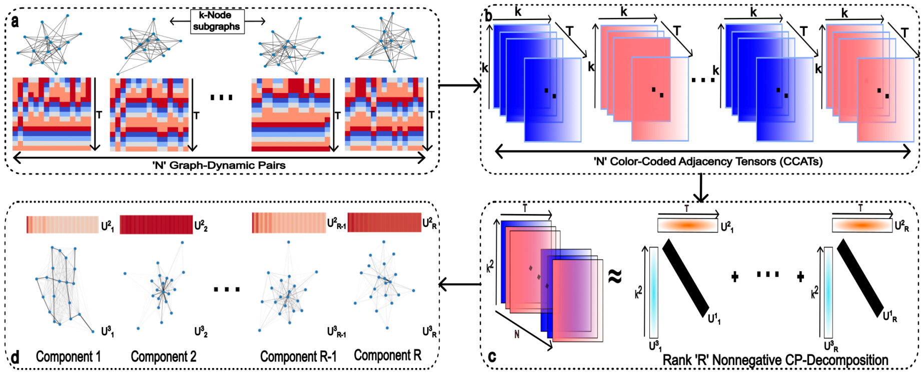

Fig. 1. Proposed Framework: (**Panel a**) Starting with N graph-dynamic pairs, we (**Panel b**) encode each of them into CCAT representations, and (**Panel c**) concatenate them after flattening into a singular tensor of shape $N \times T \times k^2$ and apply rank R NCPD on it to (**Panel d**) extract latent interpretable features from the encoded dynamics tensor

$$\mathbf{X}[i,j,t] := A_{ij} \cdot \min\{X_t(i) - X_t(j) \pmod{\kappa},\ X_t(j) - X_t(i) \pmod{\kappa}\} \tag{1}$$

$$\mathbf{X}[i,j,t] := A_{ij} \cdot |X_t(i) - X_t(j)| \cdot \mathbf{1}\left(|X_t(i) - X_t(j)| < \varepsilon\right) \tag{2}$$

where $A = (A_{ij})$ is the adjacency matrix of G, and the notation $\mathbf{1}(\cdot)$ represents the indicator function. The color-coded adjacency tensor formulation (1) is used for coupled oscillator dynamics data given its cyclic nature, and formulation (2) is used for opinion dynamics data, where ε represents the opinion difference threshold with which nodes are allowed to interact with and influence each other's opinions [13]. These color-coded adjacency tensors encompass abundant information including the network topology, evolution of dynamics, as well as the influence of connected nodes on each other.

We generate a pool of N CCATs for synchronizing and non-synchronizing dynamics observed on k-node subgraphs, sampled randomly [16] from a large parent graph (See Fig. 1**a** and 1**b**). Next, we flatten out each of these tensors into $T \times k^2$ matrices to preserve the information about the graph structure in the $k \times k$ adjacency matrices and then concatenate all N such matrices to create a final data tensor $\mathbf{Y}$ of size $N \times T \times k^2$, that encodes information about the dynamics.

Nonnegative CP Decomposition. We then apply rank R nonnegative CP decomposition on this (examples $\times$ time $\times$ graph) tensor $\mathbf{Y}$ as $\mathbf{Y} \approx \sum_{i=1}^{R} \bigotimes_{k=1}^{3} U^{(k)}[:i]$ (see Fig. 1**c**), where $U^{(1)}, U^{(2)}, U^{(3)}$ are nonnegative loading matrices with R columns and (N, T, k^2)-rows respectively, and $\bigotimes$ represents the outer product. These loading matrices represent the three sets of $R-$factors we learn, out of which the temporal factors $U_i^{(2)}$, and the graph-topology factors $U_i^{(3)}$ for $i \in \{1, \cdots, R\}$ are the most important factors for our analysis. Note that the nonnegativity constraint in this decomposition is crucial to learning an interpretable 'parts-based' representation of our input tensor [17].

The graph-topology factors can each then be reshaped back to get R matrices of size $k \times k$ which are the 'latent filters' that encode the important topological features in the underlying network which are crucial to the synchronization of the dynamics imposed on it. Next, the temporal factors represent the trend of utilization of the corresponding latent topological factors over time as the dynamics evolve (See Fig. 1**d**).

Results. We illustrate some interpretable features extracted from the **(Panel A)** FCA coupled oscillator dynamics ($\kappa = 5$), and **(Panel B)** Hegselmann-Krause (HK) opinion dynamics, on a 450−node NWS and BA networks respectively in Figure 2 using NCPD. We use rank $R = 4$, subgraph size $k = 20$, batch size $N = 2500$, and duration $T = 50$.

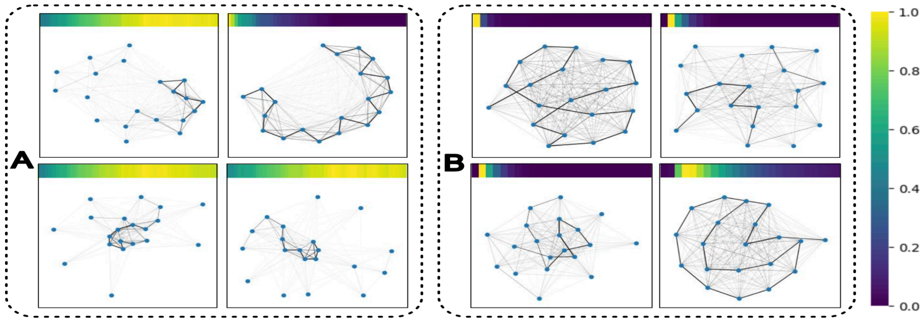

Fig. 2. Rank-4 NCPD filters for (**Panel A**) FCA dynamics and (**Panel B**) HK opinion dynamics

We observe that the latent subgraph structures learned by NCPD are representative of the parent network characteristics. In Fig. 2**A** the filters show generally sparse latent subgraphs some with cliques. This aligns with the properties of sparse NWS networks (e.g., small-world properties with high clustering coefficients). Next, in Fig. 2**B** the filters show latent subgraphs that are denser than the ones in 2**A**, with nodes that are connected to most other nodes (e.g., 'hubs'). Such hubs are characteristics of BA networks that arise due to the preferential attachment mechanism [15].

In Fig. 2**A** for the FCA dynamics, we observe that in the $(1,2)$ filter, we see a path-like pattern, and we observe that this pattern is 'transient' in the sense that its temporal utilization vanishes. This aligns with the property of FCA to synchronize relatively quickly on path-like structures for $\kappa > 3$ [12]. The other three filters seem to capture how non-synchronizing dynamics occur at the subgraph level. There, interactions due to large phase differences between nodes in the clique are persistent and are in fact utilized more strongly as the system evolves.

Next, in Fig. 2**B** for the HK dynamics, we obtain transient patterns of usage in the graph atoms, as the data tensors inherently encode network structures that represent people with opinion differences within the $\varepsilon-$threshold and thus we expect the utilization of such structures to die out eventually as these agents interact to converge towards a consensus. For example, the first atom which represents a dense pattern with a long path, and the second and third atoms where we observe a relatively sparser topology with a loose-community structure present within, are used only in the initial few iterations followed by an immediate drop in the usage. On the contrary, in the fourth atom which contains a cycle with an extended path, we observe a longer-lasting trend of utilization in the initial iterations, as seen from the prolonged activity in the color bar.

We also observe that these patterns can succinctly be captured by rank 4 decomposition and increasing the rank further leads to repetitive patterns, which implies that these subgraph-dynamic complex systems are perhaps inherently low-rank structures.

Conclusion. In summary, we show that our proposed NCPD-based framework has the potential to capture interpretable features from these large-scale complex systems that enhance our understanding of the emergent behavior and latent properties of various dynamical systems on these networks. We aim to extend our work by (1) Expanding to a wider range of dynamical systems and sizes and kinds of underlying networks, and (2) Approach this problem from a supervised perspective that would preserve the interpretability and also allow us to make predictions about the emergent properties.

Acknowledgments

The authors were partially supported by NSF grant DMS-2010035. The authors also thank the University of Wisconsin-Madison for support via the Hilldale Undergraduate/Faculty Research Fellowship.

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